

SOME REMARKS ON OSCULATING SELF-DUAL VARIETIES

SERGE LVOVSKI

ABSTRACT. Let us say that a curve $C \subset \mathbb{P}^3$ is osculating self-dual if it is projectively equivalent to the curve in the dual space $(\mathbb{P}^3)^*$ whose points are osculating planes to C . Similarly, we say that a k -dimensional subvariety $X \subset \mathbb{P}^{2k+1}$ is osculating self-dual if its second osculating space at the general point is a hyperplane and X is projectively equivalent to the variety in $(\mathbb{P}^{2k+1})^*$ whose points are second osculating spaces to X .

In this note we show that for each $k \geq 1$ there exist many osculating self-dual k -dimensional subvarieties in \mathbb{P}^{2k+1} .

1. INTRODUCTION

If $C \subset \mathbb{P}^n$ is a projective curve (not lying in a hyperplane), then its *osculating dual* is the curve $C^\vee \subset (\mathbb{P}^n)^*$ that is closure of the set of (points corresponding to) hyperplanes osculating to C . For this version of duality, the “duality theorem” $(C^\vee)^\vee = C$ in characteristic 0 also holds (see [Pie77, Theorem 5.1]).

In this note we show that there exist many curves $C \subset \mathbb{P}^3$ for which C and C^\vee are projectively equivalent: there exists a projective (linear) isomorphism $\mathbb{P}^3 \rightarrow (\mathbb{P}^3)^*$ that takes C to C^\vee . In particular, any smooth projective curve can be embedded in \mathbb{P}^3 as “osculating self-dual”.

Analog of this “osculating” duality can be defined for varieties of higher dimension as well. To wit, if $X \subset \mathbb{P}^{2k+1}$ is a k -dimensional variety such that its second osculating space at the general point is a hyperplane, then one may define $X^\vee \subset (\mathbb{P}^{2k+1})^*$ as closure of the set of points corresponding to these osculating hyperplanes; for each k , we construct a large family of k -dimensional varieties $X \subset \mathbb{P}^{2k+1}$ such that the second osculating hyperplanes at the general point is a hyperplane and X^\vee is projectively equivalent to X .

The proofs are based on the following observation: if a k -dimensional subvariety $X \subset \mathbb{P}^{2k+1}$ is Legendrian with respect to a contact structure on \mathbb{P}^{2k+1} then its second osculating space at the general point is at most $2k$ -dimensional.

1991 *Mathematics Subject Classification.* 14N99, 53D10.

Key words and phrases. Osculating space, contact structure, projective duality.

The article was prepared within the framework of a subsidy granted to the HSE by the Government of the Russian Federation for the implementation of the Global Competitiveness Program.

For $k = 2$, surfaces in \mathbb{P}^5 with four-dimensional general second osculating plane were studied by ancient Italian geometers (see [Seg07, Tog29]). In particular, Togliatti in [Tog29] classifies all non-ruled surfaces $X \subset \mathbb{P}^5$ for which $\deg X \leq 6$ and the general second osculating space is a hyperplane and essentially shows that some of these surfaces are osculating self-dual.

It should be noted that self-dual curves in \mathbb{P}^2 are much harder to construct. In particular, all known self-dual plane curves seem to have genus of normalization 0 or 1. (In the old paper [Hol26], which apparently contains many examples of self-dual plane curves, a curve C is called self-dual if several numeric invariants of C and C^* are the same, which is, of course, a weaker condition than projective equivalence.)

The paper is organized as follows. Section 3 is devoted to contact structures on \mathbb{P}^{2n-1} 's and projectivisations of cotangent bundles to \mathbb{P}^n 's, in Section 4 we construct osculating self-dual curves and varieties, and in Section 5 we show that there exist osculating self-dual varieties that cannot be obtained by the main construction of the paper. Section 2 is devoted to preliminaries.

Acknowledgements. I would like to thank Alexei Penskoi for attracting my attention to the paper [Bry82], Fyodor Zak and Nikita Kalinin for useful discussions, and Jason Starr and Robert Bryant for valuable consultations at mathoverflow.net.

2. NOTATION, CONVENTIONS, AND PRELIMINARIES

2.1. Generalities. The base field is the field \mathbb{C} of complex numbers.

Two subsets $Y_1 \subset \mathbb{P}^n$, $Y_2 \subset \mathbb{P}^n$ will be called *projectively equivalent* if there exists a projective (linear) isomorphism $F: \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $F(Y_1) = Y_2$.

If \mathcal{E} is a vector space or a vector bundle, then (closed) points of the projectivisation $\mathbb{P}(\mathcal{E})$ are lines in (the fibers of) \mathcal{E} , not hyperplanes. By $\mathbb{P}^*(\mathcal{E})$, where \mathcal{E} is a vector bundle, we mean $\mathbb{P}(\mathcal{E}^*)$; so, points of $\mathbb{P}^*(\mathcal{E})$ are hyperplanes in the fibers of \mathcal{E} .

If $L \subset \mathbb{P}(E)$ is a linear (projective) subspace, then the uniquely determined linear subspace $\hat{L} \subset E$ such that $L = \mathbb{P}(\hat{L})$ is called *deprojectivisation* of L . If $Y \subset \mathbb{P}(E)$ is a projective variety then by its deprojectivisation we mean the subvariety $\hat{Y} = \overline{\pi^{-1}(Y)} \subset E$, where $\pi: E \setminus \{0\} \rightarrow \mathbb{P}(E)$ is the canonical projection.

2.2. Contact structures. A *contact structure* on a smooth variety X is a codimension 1 subbundle \mathcal{S} of the tangent bundle \mathcal{T}_X satisfying a certain non-degeneracy condition (a precise definition can be found for example in [Kle86]; see also the sketch of proof of Proposition 3.1). If X is a variety with a contact structure \mathcal{S} , then the fiber of the vector bundle \mathcal{S} at a point $x \in X$ is denoted by $\mathcal{S}_x \subset \mathcal{T}_{x,X}$ and called *contact hyperplane at x* . Contact structures can exist only on odd-dimensional varieties.

If X is a variety with the contact structure \mathcal{S} , then a closed subvariety $Y \subset X$ is called *integral subvariety* of the structure \mathcal{S} if $\mathcal{T}_y Y \subset \mathcal{S}_y$ for each smooth point $y \in Y$.

Locally, each contact structure on X can be defined as (closure of) the family of hyperplanes in tangent spaces that are kernels of a nonvanishing 1-form $\omega \in \Gamma(\Omega_U^1)$, where $U \subset X$ is a Zariski open set (globally this ω is a section of $\Omega_X^1 \otimes (\mathcal{T}_X/\mathcal{S})$). A subvariety $Y \subset X$ is integral if and only if the restriction of ω to its smooth part is zero.

If X is a variety with a contact structure, $\dim X = 2n - 1$, then integral subvarieties of (maximal possible) dimension $n - 1$ are called *Legendrian subvarieties* of X with respect to this contact structure.

2.3. Osculating spaces and osculating duality. If $X \subset \mathbb{P}^N$ is a projective variety and $x \in X$ is a non-singular point, one says that a hyperplane $H \subset \mathbb{P}^n$ *osculates at x to order s* if $H \ni x$ and the local equation of $H \cap X$ in the local ring $\mathcal{O}_{x,X}$ lies in \mathfrak{m}_x^{s+1} , where $\mathfrak{m}_x \subset \mathcal{O}_{x,X}$ is the maximal ideal. Analytically this means the following: if z_1, \dots, z_n are analytic local coordinates on X near x and f is a local equation of H at x , then the power series expansion of f begins with terms of degree $\geq s + 1$. A hyperplane osculates at x to order 1 if and only if H is tangent to X at x .

The intersection of all hyperplanes osculating to X at x to order s is called *s 'th osculating space to X at x* and denoted by $\text{Osc}_x^s X$ (if no hyperplane osculates to order s at x , we assume that $\text{Osc}_x^s X$ coincides with the linear span of X). The space $\text{Osc}_x^1 X \subset \mathbb{P}^N$ is nothing but the embedded tangent space $T_x X \subset \mathbb{P}^n$.

If $X \subset \mathbb{P}^n$ is a curve that is not contained in a hyperplane, then for general $x \in X$ one has $\dim \text{Osc}_x^j X = j$ for $1 \leq j \leq n - 1$, and $\text{Osc}_x^{n-1} X$ is exactly the osculating hyperplane as defined in the introduction.

If $x \in X$ is a non-singular point and H is a tangent hyperplane to X at x , then the image of the local equation of $H \cap X$ in $\mathfrak{m}_x^2/\mathfrak{m}_x^3$ defines (up to a multiplicative constant) an element of $\text{Sym}^2(\mathfrak{m}_x/\mathfrak{m}_x^2) = \text{Sym}^2(\mathcal{T}_x X)^*$. All the symmetric bilinear forms in the tangent space (with all their multiples) corresponding, via the procedure above, to hyperplanes $H \supset T_x X$, form a linear subspace of $\text{Sym}^2 \mathcal{T}_x X$. This linear space is called *second fundamental form of x at X* ; it will be denoted $\Phi_x^2(X)$. If we fix local analytic coordinates z_1, \dots, z_n at x , then elements of $\Phi_x^2(X)$ are polynomials of degree s in dz_1, \dots, dz_n ; abusing the language, we will write them as polynomials in z_1, \dots, z_n . One has $\dim \Phi_x^2(X) = \dim \text{Osc}_x^2 X - \dim X$.

Suppose that $X \subset \mathbb{P}^{2n-1}$ is a projective variety of dimension $n - 1$ that is not contained in a hyperplane. The expected value of $\dim \text{Osc}_x^2 X$ for general $x \in X$ is $2n - 1$, i.e., in the general case this osculating space coincides with the ambient \mathbb{P}^{2n-1} . If, however, for general x , $\text{Osc}_x^2 X$ is a hyperplane, or, equivalently, $\dim \Phi_x^2(X) = n - 1 = \dim X$ (if $\dim X = 1$, this is automatic, otherwise it is a non-trivial condition), we denote by $X^\vee \subset (\mathbb{P}^{2n-1})^*$ the closure of the set of hyperplanes $\text{Osc}_x^2(X)$ for general $x \in X$.

(in the paper [Val06] this variety is denoted by $X^{3\vee}$). The subvariety $X^\vee \subset (\mathbb{P}^n)^*$ will be called *osculating dual* to X . If X is projectively equivalent to X^\vee , we will say that X is *osculating self-dual*.

3. WELL-KNOWN CONTACT STRUCTURES ON \mathbb{P}^{2n-1} AND $\mathbb{P}^*(\mathcal{T}\mathbb{P}^n)$

If E is a vector space of even dimension $2n$ and B is a non-degenerate skew-symmetric bilinear form on E , we define a contact structure on $\mathbb{P}^{2n-1} = \mathbb{P}(E)$ as follows. If $x = (v)$ is a point of $\mathbb{P}(E)$, where $v \in E \setminus \{0\}$, then the contact hyperplane $\mathcal{S}_x \subset \mathcal{T}_x\mathbb{P}(E)$ is $\mathcal{T}_x\mathbb{P}(v^\perp)$, where $v^\perp \subset E$ is the skew-orthogonal complement to v with respect to B . If in coordinates the form B is defined by the formula

$$(3.1) \quad B((z_0, \dots, z_{2n}), (w_0, \dots, w_{2n})) = \sum_{i=0}^{n-1} (z_{2i}w_{2i+1} - z_{2i+1}w_{2i}),$$

then on the affine open set $\{(1 : z_1 : \dots : z_{2n-1}) \in \mathbb{P}^{2n-1}\}$ the contact structure corresponding to B can be defined by the form

$$(3.2) \quad \omega = dz_1 + \sum_{i=1}^{n-1} (z_{2i}dz_{2i+1} - z_{2i+1}dz_i).$$

Since any two non-degenerate skew-symmetric forms are equivalent, any two contact structures on \mathbb{P}^{2n-1} obtained by the above construction are mapped to each other by a projective automorphism of \mathbb{P}^{2n-1} .

Actually, there are no other contact structures on projective spaces. The following proposition seems to belong to folklore.

Proposition 3.1. *Any contact structure on $\mathbb{P}^{2n-1} = \mathbb{P}(E)$, $\dim E = 2n$, corresponds to a non-degenerate skew-symmetric bilinear form on E .*

Sketch of proof. If a contact structure is defined by a subbundle $\mathcal{S} \subset \mathcal{T}_{\mathbb{P}(E)}$, put $\mathcal{L} = \mathcal{T}_{\mathbb{P}(E)}/\mathcal{S}$; \mathcal{L} is an invertible sheaf. The mapping

$$(s_1, s_2) \mapsto [s_1, s_2] \bmod \mathcal{S},$$

where s_1 and s_2 are local sections of $\mathcal{T}_{\mathbb{P}(E)}$ and the brackets stand for commutator of vector fields, is a sheaf homomorphism $\mathcal{S} \otimes_{\mathcal{O}_{\mathbb{P}(E)}} \mathcal{S} \rightarrow \mathcal{L}$ which factors through $\bigwedge^2 \mathcal{S}$; one of the equivalent definitions of contact structures is that \mathcal{S} is a contact structure if and only if the resulting homomorphism $\psi: \bigwedge^2 \mathcal{S} \rightarrow \mathcal{T}_{\mathbb{P}(E)}/\mathcal{S} = \mathcal{L}$ is a non-degenerate skew-symmetric form on \mathcal{S} with values in \mathcal{L} . Since ψ is non-degenerate, it induces an isomorphism

$$(3.3) \quad \mathcal{S} \xrightarrow{\sim} \mathcal{S}^* \otimes \mathcal{L}.$$

Since $c_1(\mathcal{S}) = c_1(\mathcal{T}_{\mathbb{P}(E)}) - c_1(\mathcal{L})$ and $c_1(\mathcal{S}^* \otimes \mathcal{L}) = -c_1(\mathcal{S}) + (2n-2)c_1(\mathcal{L})$, the isomorphism (3.3) implies that $2nc_1(\mathcal{L}) = 2c_1(\mathcal{T}_{\mathbb{P}(E)})$, whence $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(E)}(2)$. The epimorphism $\pi: \mathcal{T}_{\mathbb{P}(E)} \rightarrow \mathcal{L} = \mathcal{O}_{\mathbb{P}(E)}(2)$ is a section of $\Omega_{\mathbb{P}(E)}^1(2)$; it follows from the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}(E)}^1(2) \rightarrow E^* \otimes \mathcal{O}_{\mathbb{P}(E)} \rightarrow \mathcal{O}_{\mathbb{P}(E)}(2) \rightarrow 0$$

(Euler's sequence twisted by $\mathcal{O}(2)$) that $H^0(\Omega_{\mathbb{P}(E)}^1(2)) = \bigwedge^2 E^*$. Now the element of $\bigwedge^2 E^*$ corresponding to π is the desired skew-symmetric form. \square

The reader may ignore this proposition and further on, every time we mention a contact structure on \mathbb{P}^{2n-1} , substitute “a contact structure corresponding to a skew-symmetric form” instead.

Proposition 3.2. *Suppose that E is an even-dimensional vector space of dimension $2n$ endowed with a non-degenerate skew-symmetric bilinear form B and that $Y \subset \mathbb{P}(E)$ is a projective subvariety. Then the following two assertions are equivalent.*

(1) *Y is integrable with respect to the contact structure corresponding to the form B .*

(2) *For any smooth point $y \in Y$, the deprojectivisation $\widehat{T_y Y} \subset E$ of the embedded tangent space $T_y Y \subset \mathbb{P}(E)$ is isotropic with respect to the form B .*

Proof. The implication (2) \Rightarrow (1) is very easy. To wit, if $y = (v)$, where $v \in F = \widehat{T_y Y}$, then $B(v, w) = 0$ for any $w \in F$ since F is isotropic, so $F \subset v^\perp$ and $T_y Y \subset T_y \mathbb{P}(E)$ is contained in the contact hyperplane at the point y .

To prove the implication (1) \Rightarrow (2) denote by $\pi: E \setminus \{0\} \rightarrow \mathbb{P}(E)$ the natural projection. Since π is submersive, the pullback $\pi^* \mathcal{S} \subset \mathcal{T}_{E \setminus \{0\}}$ is a subbundle of codimension 1, where \mathcal{S} is the contact structure corresponding to B . If the bilinear form B is defined by the formula (3.1), then the family of hyperplanes in tangent spaces defined by the subbundle $\pi^* \mathcal{S}$ is the family of kernels of the 1-form $\eta = \sum_{i=0}^{2n-1} z_{2i} dz_{2i+1}$. Since Y is an integral variety of \mathcal{S} , the form η vanishes on \hat{Y}_{sm} , whence $d\eta|_{\hat{Y}_{\text{sm}}} = 0$. Since $d\eta = \sum_{j=0}^{2n-1} dz_{2j} \wedge dz_{2j+1}$, this vanishing is equivalent to the assertion that tangent spaces to Y_{sm} are isotropic with respect to B . Since these tangent spaces are deprojectivisations of embedded tangent spaces to Y , we are done. \square

Proposition 3.3. *Suppose that $\mathbb{P}^{2n-1} = \mathbb{P}(E)$ is endowed with the contact structure corresponding to a non-degenerate skew-symmetric bilinear form B on E and that $Y \subset \mathbb{P}^{2n-1}$ is a Legendrian projective subvariety with respect to this contact structure. If Y is contained in a hyperplane in \mathbb{P}^{2n-1} , then Y is a cone over a variety $Y_1 \subset \mathbb{P}^{2n-3} = \mathbb{P}(E_1)$, where $E_1 \subset E$ is a linear subspace of codimension 2; besides, the restriction of the form B to E_1 is non-degenerate and the variety $Y_1 \subset \mathbb{P}(E_1)$ is Legendrian with respect to the contact structure corresponding to restriction $B|_{E_1}$.*

Proof. Suppose that a Legendrian (with respect to B) projective subvariety $Y \subset \mathbb{P}(E)$ lies in a hyperplane $H \subset \mathbb{P}(E)$. One has $H = \mathbb{P}(v^\perp)$ for some $v \in E \setminus \{0\}$. Since the form B is non-degenerate, there exists a $(2n - 2)$ -dimensional linear subspace $E_1 \subset v^\perp$ and a vector $w \in E_1^\perp$ such that $B(v, w) \neq 0$ and $E = E_1 \oplus \langle v, w \rangle$ is a skew-orthogonal direct sum; the restriction of B to E_1 is again non-degenerate.

Put $p = (v) \in \mathbb{P}(E)$ and denote by $\pi_p: H \dashrightarrow \mathbb{P}(E_1)$ the projection from p . If $L \subset H$ is a linear subspace such that the deprojectivisation \hat{L} is isotropic, then the deprojectivisation of $\pi_p(L)$ in $\mathbb{P}(E_1)$ is also isotropic. Put $Y_1 = \pi_p(Y)$. It follows from Proposition 3.2 that deprojectivisations of tangent spaces to smooth points of Y are isotropic; now Sard's theorem together with the above observation implies that deprojectivisations of tangent spaces at almost all points of Y_1 are also isotropic.

If $\dim Y_1 = \dim Y = n - 1$, we obtain a contradiction since in that case dimension of these deprojectivisations is $n > 2(n - 1)/2$. Thus, $\dim Y_1 = \dim Y - 1 = n - 1$ and Y is a cone over Y_1 with vertex p . Finally, since (deprojectivisations of) tangent spaces to Y_1 are isotropic, the subvariety $Y_1 \subset \mathbb{P}(E_1)$ is Legendrian with respect to the restriction of B to E_1 . \square

Corollary 3.4. *Suppose that C is a projective Legendrian curve in \mathbb{P}^3 with a contact structure. If C is contained in a plane, then C is a line.* \square

If X is a smooth variety, one can define a canonical contact structure on $V = \mathbb{P}^*(\mathcal{T}_X)$ as follows. If $p = (x, H) \in V$, where $x \in X$ and $H \subset \mathcal{T}_x X$ is a hyperplane, then $\mathcal{S}_p = \pi_*^{-1}(H) \subset \mathcal{T}_p V$, where $\pi: V \rightarrow X$ is the projection and π_* is the derivative of π .

If $Y \subset X$ is a subvariety, then it is easy to check that

$$\mathcal{P}_Y = \overline{\{(y, H) \in V = \mathbb{P}^*(\mathcal{T}_X) : y \in Y_{\text{smooth}}, H \supset \mathcal{T}_y Y\}}$$

is a Legendrian subvariety of V . It follows from Sard's theorem that any Legendrian subvariety of V has the form \mathcal{P}_Y for some $Y \subset X$ (see [Kle86]). We will say that \mathcal{P}_Y is the *conormal variety* of Y .

We will be using the above construction for $X = \mathbb{P}^n$. In this situation $\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ is just the incidence relation:

$$\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n}) = \{(x, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^* : x \in H\}.$$

In coordinates, if $(x_0 : \dots : x_n)$ are homogeneous coordinates in \mathbb{P}^n and $(y_0 : \dots : y_n)$ are the dual homogeneous coordinates in $(\mathbb{P}^n)^*$, one has

$$(3.4) \quad \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n}) = \{((x_0 : \dots : x_n), (y_0 : \dots : y_n)) \in \mathbb{P}^n \times (\mathbb{P}^n)^* : \sum x_i y_i = 0\}.$$

Thus, $\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ is a hyperplane section of Segre variety $\mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n^2+2n}$. When referring to $\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ as projective variety we will always mean this embedding $\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n}) \hookrightarrow \mathbb{P}^{n^2+2n-1}$.

The following result is due essentially to R. Bryant, at least for $n = 2$ (see [Bry82, proof of Theorem F]).

Proposition 3.5. *Suppose that \mathbb{P}^{2n-1} , with homogeneous coordinates $(z_0 : \dots : z_{2n-1})$, is endowed with the contact structure corresponding to the skew-symmetric form (3.1) and that $\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ defined by the equation (3.4) is endowed with the canonical contact structure. Then the rational mapping $\vartheta: \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n}) \dashrightarrow \mathbb{P}^{2n-1}$ defined by the formula*

$$\vartheta: ((x_0 : \dots : x_n), (y_0 : \dots : y_n)) \mapsto (z_0 : \dots : z_{2n-1}),$$

where

$$\begin{aligned}
 (3.5) \quad & z_0 = x_0 y_1, \\
 & z_1 = \frac{1}{2}(x_1 y_1 - x_0 y_0), \\
 & z_{2k-2} = x_k y_1, \quad 2 \leq k \leq n, \\
 & z_{2k-1} = -\frac{1}{2}x_0 y_k, \quad 2 \leq k \leq n,
 \end{aligned}$$

is a birational isomorphism that agrees with the named contact structures. This birational isomorphism induces an isomorphism between the Zariski open subsets

$$(3.6) \quad V = \{(x_0 : \dots : x_n), (y_0 : \dots : y_n) : x_0 \neq 0, y_1 \neq 0\} \subset \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$$

and

$$W = \{(z_0 : z_1 : \dots : z_{2n-1}) : z_0 \neq 0\} \subset \mathbb{P}^{2n-1}.$$

Proof. A straightforward check shows that the rational mapping $\beta: \mathbb{P}^{2n-1} \dashrightarrow \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ defined by the formulas

$$\begin{aligned}
 x_0 &= z_0^2, & y_0 &= -z_0 z_1 + \sum_{j=1}^{n-1} z_{2j} z_{2j+1}, \\
 x_1 &= z_0 z_1 + \sum_{j=1}^{n-1} z_{2j} z_{2j+1}, & y_1 &= z_0^2, \\
 x_k &= z_0 z_{2k-2}, & y_k &= -2z_0 z_{2k-1}, \quad 2 \leq k \leq n
 \end{aligned}$$

is inverse to ϑ and that ϑ performs an isomorphism between V and W . To check that ϑ agrees with the contact structures it suffices to show that its inverse β agrees with contact forms on some non-empty Zariski open set. Put

$$V_1 = \{((x_0 : \dots : x_n), (y_0 : \dots : y_n)) \in \mathbb{P}^n \times (\mathbb{P}^n)^* : x_0 \neq 0, y_0 \neq 0, y_1 \neq 0\};$$

we may and will assume that $x_0 = y_0 = 1$ on $V_1 \subset V$. For each j , $2 \leq j \leq n$, put $\xi_j = y_j/y_1$. Then $(x_1, \dots, x_n, \xi_2, \dots, \xi_n)$ are local coordinates on V_1 , and it is easy to see that in these (x, ξ) coordinates on V_1 the canonical contact structure on $\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ may be defined as the family of kernels of the form

$$(3.7) \quad \eta = dx_1 + \xi_2 dx_2 + \dots + \xi_n dx_n.$$

An immediate check shows that $\beta^* \eta = \omega$, where η is defined by (3.7) and ω is defined by (3.2), so we are done. \square

Proposition 3.6. *The birational isomorphism $\vartheta: \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n}) \rightarrow \mathbb{P}^{2n-1}$ is induced by a projection $\pi_L: \mathbb{P}^{n^2+2n-1} \dashrightarrow \mathbb{P}^{2n-1}$, where $L \subset \mathbb{P}^{n^2+2n-1}$ is a linear subspace of dimension $n^2 - 1$. The intersection $L \cap \mathbb{P}^{n^2+2n-1}$ has the form*

$$L \cap \mathbb{P}^{n^2+2n-1} = \{(x, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^* : x \in H_0, H \ni x_0, x \in H\},$$

where $H_0 \subset \mathbb{P}^n$ is the hyperplane defined by the equation $x_0 = 0$ and $p_0 \in H_0$ is the point with homogeneous coordinates $(0 : 1 : 0 : \dots : 0)$.

Proof. Proposition 3.5 shows that ϑ is defined by bihomogeneous in x 's and y 's polynomials of bidegree $(1, 1)$, so it is a projection of (a hyperplane section of) Serge variety, with center L of dimension $(n^2 + 2n - 1) - (2n - 1) - 1$. The intersection $L \cap \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ is the set of points where all the polynomials in the right-hand sides of (3.5) vanish; it is easy to see that this happens if and only if $x_0 = y_1 = 0$, which implies the proposition. \square

4. CONSTRUCTION OF SELF-DUAL VARIETIES

Throughout this section, $\mathbb{P}^{2n-1} = \mathbb{P}(E)$, where the $2n$ -dimensional linear space E is endowed with a non-degenerate skew-symmetric bilinear form B . By contact structure on \mathbb{P}^n we will mean the contact structure associated with B . If $p = (v) \in \mathbb{P}^{2n-1}$, where $v \in E \setminus \{0\}$, then by $p^\perp \subset \mathbb{P}^{2n-1}$ we mean $\mathbb{P}((v)^\perp)$. Obviously, p is contained in the hyperplane p^\perp .

Proposition 4.1. *If $X \subset \mathbb{P}^{2n-1}$ is an integral subvariety with respect to a contact structure, then $\text{Osc}_p^2 X \subset p^\perp$ for any smooth $p \in X$.*

Proof. Put $\dim X = d$. Suppose that $p \in X$ is a smooth point and x_1, \dots, x_n are local analytic coordinates near x . Locally (in the classical topology) near p the variety $X \subset \mathbb{P}^n$ can be parametrized by the formula

$$(x_1, \dots, x_d) \mapsto (v(x_1, \dots, x_d)),$$

where $v: U \rightarrow E \setminus \{0\}$ is a holomorphic immersion and $U \subset \mathbb{C}^d$ is an open set; the subspace $\text{Osc}_p^k X \subset \mathbb{P}(E)$ is projectivisation of the linear space spanned by v and all its partial derivatives up to the order k .

Since X is an integral variety of the contact structure associated with B , one has

$$(4.1) \quad B(x, \partial v / \partial x_i) = 0, \quad 1 \leq i \leq d$$

by definition of the contact structure corresponding to B and

$$(4.2) \quad B(\partial v / \partial x_i, \partial v / \partial x_j) = 0, \quad 1 \leq i, j \leq d$$

by Proposition 3.2. Differentiating (4.1) with respect to x_i , one has

$$B(\partial v / \partial x_i, \partial v / \partial x_i) + B(v, \partial^2 v / \partial x_i^2) = B(v, \partial^2 v / \partial x_i^2) = 0,$$

so $\partial^2 v / \partial x_i^2 \in (v)^\perp$. Differentiating (4.1) with respect to x_j , $j \neq i$, one has

$$B(\partial v / \partial x_j, \partial v / \partial x_i) + B(v, \partial^2 v / \partial x_i \partial x_j) = B(v, \partial^2 v / \partial x_i \partial x_j) = 0$$

(the first summand vanishes by virtue of (4.2)), so $\partial^2 v / \partial x_i \partial x_j \in (v)^\perp$. Thus, $\text{Osc}_p^2 X \subset p^\perp$ as required. \square

Proposition 4.2. *Suppose that $X \subset \mathbb{P}^{2n-1}$ is a Legendrian subvariety with respect to a contact structure. If $\dim \text{Osc}_p^2 X = 2n - 2$ for general $p \in X$ and X is not contained in a hyperplane, then X is osculating self-dual.*

Proof. Proposition 4.1 implies that $\text{Osc}_p^2 X \subset p^\perp$; since dimensions are the same, these linear spaces coincide. Now the desired linear isomorphism $\mathbb{P}^n \rightarrow (\mathbb{P}^n)^*$ that maps X to X^\vee is the one induced by the isomorphism $E \rightarrow E^*$ corresponding to the bilinear form B . \square

Lemma 4.3. *Suppose that $X \subset \mathbb{P}^n$ is an irreducible subvariety. If X is not contained in a hyperplane and X is not a cone then the conormal variety $\mathcal{P}_X \subset \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ has a non-empty intersection with the Zariski open subset $V \subset \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ defined in (3.6).*

In particular, if $n = 2$ then $\mathcal{P}_X \cap V \neq \emptyset$ provided that X is not a line.

Proof. Since X is not contained in a hyperplane, $X \cap \mathbb{A}^n = \{(1 : x_1 : \dots : x_n)\} \neq \emptyset$. Thus, to check that $\mathcal{P}_X \cap V \neq \emptyset$ it suffices to check that the coordinate y_1 is not identically zero on \mathcal{P}_X . Assume the converse; then all the tangent hyperplanes to smooth points of X pass through the point $p = (0 : 1 : 0 : \dots : 0)$, which is possible only if X is a cone with vertex p (to justify this assertion, apply Sard's theorem to the projection with center p), which contradicts the hypothesis. \square

Proposition 4.4. *If $X \subset \mathbb{P}^n$, $n \geq 3$, is a general hypersurface of degree at least 3 and $\mathcal{P}_X \subset \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n})$ is its conormal variety, then the proper image $C = \vartheta(\mathcal{P}_X) \subset \mathbb{P}^{2n-1}$, where $\vartheta : \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^n}) \rightarrow \mathbb{P}^{2n-1}$ is the birational isomorphism defined in Proposition 3.5, is an $(n-1)$ -dimensional subvariety such that $\dim \text{Osc}_x^2 C = 2n-2$ for general $x \in C$, C is not contained in a hyperplane, and C^\vee is projectively equivalent to C .*

Proof. Since we may assume that X is not a cone, Lemma 4.3 shows that the image $C = \vartheta(\mathcal{P}_X)$ is well defined. Since the birational isomorphism ϑ agrees with the contact structures, the variety C is Legendrian with respect to a contact structure on \mathbb{P}^{2n-1} . Now Proposition 4.2 shows that to prove the proposition it suffices to check that, for general X , the variety C is not contained in a hyperplane and $\dim \text{Osc}_x^2 C = 2n-2$ for general $x \in C$.

Since dimensions of osculating spaces are lower semicontinuous and dimension of the second osculating space to a Legendrian subvariety in \mathbb{P}^{2n-1} is at most $2n-2$ (Proposition 4.1), the second assertion will follow once we have, for each $d \geq 3$, an example of a hypersurface $X \subset \mathbb{P}^n$ of degree d such that the general osculating space to $\vartheta(\mathcal{P}_X)$ has dimension $2n-2$. Let us look for such examples among hypersurfaces with the equation $x_0^{d-1}x_1 + F(x_2, \dots, x_n) = 0$, where $(x_0 : \dots : x_n)$ are homogeneous coordinates in \mathbb{P}^n and F is a homogeneous polynomial of degree d . On the affine open subset $\{x_0 = 1\}$ this hypersurface has equation $x_1 + F(x_2, \dots, x_n) = 0$.

It follows from Proposition 3.5 that $\vartheta(\mathcal{P}_X)$ is closure of the set of points $(1 : z_1 : \dots : z_{2n-1})$, where

$$(4.3) \quad \begin{aligned} z_1 &= \frac{2-d}{2} F(x_2, \dots, x_n), \\ z_{2k-2} &= x_k, \quad 2 \leq k \leq n, \\ z_{2k-1} &= -\frac{1}{2} \cdot \frac{\partial F}{\partial x_k}, \quad 2 \leq k \leq n \end{aligned}$$

(x_2, \dots, x_n are arbitrary).

The second fundamental form is spanned by Hessians of right-hand sides of (4.3), where by Hessian of a function φ depending on the variables x_2, \dots, x_n we mean the quadratic form $\sum_{i,j} (\partial^2 \varphi / \partial x_i \partial x_j) t_i t_j$. Now if we put $F = x_2^d + \dots + x_n^d$, then it is easy to check that these Hessians span an $(n-1)$ -dimensional space, as required.

It remains to check that for the general X the variety C is not contained in a hyperplane. To that end, we invoke Proposition 3.3. According to this proposition, if C is contained in a hyperplane, then C is a cone over $C_1 \subset \mathbb{P}^{2n-3} \subset \mathbb{P}^{2n-1}$, where C_1 is Legendrian with respect to a contact structure on \mathbb{P}^{2n-3} . Since C is a cone over C_1 , dimensions of the second fundamental form at the general point are the same for C and C_1 ; since C_1 is Legendrian in \mathbb{P}^{2n-3} , $\dim \text{Osc}_x^2 C_1 \leq n-2$ by Proposition 4.1, so $\dim \text{Osc}_x^2 C \leq n-2$ as well, but we know that this is not the case for $C = \vartheta(X)$ for general X . This contradiction completes the proof. \square

In the case of curves in \mathbb{P}^3 one can say a bit more.

Proposition 4.5. *Suppose that $X \subset \mathbb{P}^2$ is an irreducible projective curve of degree greater than one. Then there exists an osculating self-dual curve $C \subset \mathbb{P}^3$ such that C is birational to X and $\deg C = \deg X + \deg X^*$, where $X^* \subset (\mathbb{P}^2)^*$ is the dual curve.*

Proof. Arguing as in the proof of Proposition 4.4, if $\mathcal{P}_X \subset \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^2})$ is the conormal variety of X , then $C = \vartheta(\mathcal{P}_X) \subset \mathbb{P}^3$ is Legendrian, hence self-dual, if C is not contained in a plane and $\mathcal{P}_X \cap W \neq \emptyset$ (since $\dim \text{Osc}_p^2 C = 2$ automatically for general $p \in C$ if $C \subset \mathbb{P}^3$ is a curve that is not contained in a plane). Since $X \subset \mathbb{P}^2$ is not a line, it is not a cone, so Lemma 4.3 ensures that $\mathcal{P}_X \cap W \neq \emptyset$ and $C = \vartheta(\mathcal{P}_X)$ is well defined. To be able to control $\deg C$, recall that, according to Proposition 3.6, the rational mapping ϑ is induced by the projection $\pi_L: \mathbb{P}^7 \dashrightarrow \mathbb{P}^3$, where $L \subset \mathbb{P}^7$ is a 3-dimensional linear space such that

$$L \cap \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^3}) = \{(x, H) \in \mathbb{P}^2 \times (\mathbb{P}^2)^*: x \in H_0, H \ni x_0, x \in H\};$$

here, $H_0 \subset \mathbb{P}^2$ is a line and $p_0 \in H_0$ is a point. So, this intersection is the union of two lines, $\{(x, H_0): x \in H_0\}$ and $\{(x_0, H): H \ni x_0\}$; these lines intersect at the point (x_0, H_0) . It is clear that $L \cap \mathcal{P}_X$ is the set of couples (x, H) such that either $H = H_0$ is tangent to X at the point x or H is

tangent to X at the point $x = x_0$ (if $x \in X$ is singular, we say that a line H is tangent to C at X if it is a limit of tangents at smooth points tending to x). Thus, if X is not tangent to H_0 (i.e., $H_0 \notin X^*$) and $x_0 \notin X$, then $\mathcal{P}_X \cap L = \emptyset$ since the restriction of the projection $\pi_L = \vartheta$ to the subset $V = \{((x_0 : x_1 : x_2), (y_0 : y_1 : y_2)) \in \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^2}) : x_0 \neq 1, y_1 \neq 1\}$ is an isomorphism onto its image (see Proposition 3.5), this implies that $\deg C = \deg \mathcal{P}_X$, where in the right-hand side we regard \mathcal{P}_X as a curve in the \mathbb{P}^7 in which $\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^2})$ is embedded. Denoting the projections of $\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^2})$ on \mathbb{P}^2 and $(\mathbb{P}^2)^*$ by pr_1 and pr_2 respectively, one sees that $\mathcal{O}_{\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^2})}(1) = \text{pr}_1^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \text{pr}_2^* \mathcal{O}_{(\mathbb{P}^2)^*}(1)$. So, if $L \cap \mathcal{P}_X = \emptyset$, then

$$\deg C = \deg \mathcal{O}_{\mathbb{P}^*(\mathcal{T}_{\mathbb{P}^2})}(1)|_{\mathcal{P}_X} = \deg X + \deg X^*.$$

It remains to observe that for any curve $X \subset \mathbb{P}^2$ there exists a curve $X_1 \subset \mathbb{P}^2$ such that X_1 is projectively equivalent to X and X_1 is not tangent to H_0 and does not pass through p_0 . Putting $C = \vartheta(\mathcal{P}_{X_0})$ one obtains the required self-dual curve. \square

Proposition 4.6. *Any smooth projective curve is isomorphic to a curve in \mathbb{P}^3 that is Legendrian with respect to a contact structure.*

In particular, any smooth projective curve can be embedded in \mathbb{P}^3 as an osculating self-dual curve.

We begin the proof with two lemmas. In these lemmas we assume that homogeneous coordinates on \mathbb{P}^3 are $(z_0 : z_1 : z_2 : z_3)$, homogeneous coordinates on \mathbb{P}^2 are $(x_0 : x_1 : x_2)$ and dual homogeneous coordinates on $(\mathbb{P}^2)^*$ are $(y_0 : y_1 : y_2)$. By tangent line to a plane nodal curve X we mean a line that is tangent either to C at a smooth point or to a branch of C at a node. By $\vartheta : \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^2}) \subset \mathbb{P}^3$ we mean the birational morphism defined in Proposition 3.5.

Lemma 4.7. *Suppose that $X \subset \mathbb{P}^2$ is a nodal curve with the following properties.*

- (1) *X intersects transversally the line $\{x_0 = 0\}$.*
- (2) *The lines tangent to X at its intersection points with the line $\{x_0 = 0\}$, do not pass through the point $(0 : 1 : 0)$.*
- (3) *The lines tangent to X at inflection points (including inflection points of branches at nodes, if such nodes exist) do not pass through the point $(0 : 1 : 0)$.*
- (4) *The curve X does not pass through the point $(0 : 0 : 1)$.*

Then the conormal variety $\mathcal{P}_X \subset \mathbb{P}^(\mathcal{T}_{\mathbb{P}^2})$ is smooth, lies in the open set where the birational isomorphism ϑ is defined, and the mapping $\vartheta|_{\mathcal{P}_X} : \mathcal{P}_X \rightarrow \mathbb{P}^3$ is an immersion.*

Proof. Since X is nodal, \mathcal{P}_X is smooth, so we are only to check that ϑ is defined on \mathcal{P}_X and that the derivative of the restriction $\vartheta|_{\mathcal{P}_X}$ does not vanish.

The first assertion follows from Proposition 3.6 and hypothesis (2).

To check that $\vartheta|_{\mathcal{P}_X}$ is an immersion observe that the restriction of ϑ to the subset $V = \{((x_0 : x_1 : x_2), (y_0 : y_1 : y_2)) \in \mathbb{P}^*(\mathcal{T}_{\mathbb{P}^2}) : x_0 \neq 0, y_0 \neq 0\}$ is an isomorphism onto its image. Thus, it suffices to check that $\vartheta|_{\mathcal{P}_X}$ is an immersion at the points of \mathcal{P}_X for which either $x_0 = 0$ or $y_1 = 0$ (these coordinates cannot both vanish for a point of \mathcal{P}_X because of hypothesis (2)).

Case 1. Points of \mathcal{P}_X for which $x_0 \neq 0, y_1 = 0$. If $x_0 \neq 0$ and $y_1 = 0$ for a point $(p, H) \in \mathcal{P}_X$, then the curve X (which is smooth at p thanks to hypothesis (1)) can be analytically parametrized near p by the formula $\gamma : t \mapsto (1 : a(t) : b(t))$, where $\gamma(0) = p$ and $b'(0) = 0$. Since the tangent line at $p(t)$ has equation

$$(a'(t)b(t) - b'(t)a(t))x_0 + b'(t)x_1 - a'(t)x_2 = 0,$$

the curve \mathcal{P}_X can be parametrized near (p, H) as

$$t \mapsto ((1 : a(t) : b(t)), (a'(t)b(t) - b'(t)a(t) : b'(t) : -a'(t)));$$

using formula (3.5) we see that the curve $\vartheta(\mathcal{P}_X)$ near the point (p, H) can be parametrized by

$$v : t \mapsto (b'(t) : (2a(t)b'(t) - a'(t)b(t))/2 : b(t)b'(t) : a'(t)/2).$$

The mapping v is not an immersion exactly at the points where v and v' are proportional; taking into account the equation $b'(0) = 0$, one has

$$(4.4) \quad v'(0) = (b''(0) : * : * : *), \quad v(0) = (0 : * : * : a'(0)/2),$$

where stars stand for irrelevant terms. It follows from (4.4) that if $v(0)$ and $v'(0)$ are proportional then $a'(0)b''(0) = 0$. However, $a'(0) \neq 0$ since $b'(0) = 0$. Thus, $b''(0) = 0$, so p is an inflexion point of the curve X , and the tangent to X at p passes through $(0 : 1 : 0)$ since $b'(0) = 0$, which contradicts hypothesis (3). Thus, $\vartheta|_{\mathcal{P}_X}$ is an immersion at (p, H) .

Case 2. Points of \mathcal{P}_X for which $x_0 = 0, y_1 \neq 0$. If $x_0 = 0$ and $y_1 \neq 0$ for a point $(p, H) \in \mathcal{P}_X$, then $x_1 \neq 0$ for the point p thanks to hypothesis (4). So, near the point p the curve X (or, if p is a node, the branch to which H is tangent) can be parametrized as $t \mapsto (a(t) : 1 : b(t))$, where $a(0) = 0$. A computation similar to what we did in Case 1 shows that \mathcal{P}_X near the point (p, H) can be parametrized as

$$t \mapsto ((a(t) : 1 : b(t)), (b'(t) : a(t)'b(t) - a(t)b'(t) : -a'(t)));$$

and the curve $\vartheta(\mathcal{P}_X)$ near the point (p, H) can be parametrized as

$$v : t \mapsto (a(t)a'(t)b(t) - (a(t))^2b'(t) : (a'(t)b(t) - 2a(t)b'(t))/2 : a'(t)b(t)^2 - a(t)b(t)b'(t) : a(t)a'(t)/2),$$

Again v fails to be immersion where v and v' are proportional. Taking into account the equation $a(0) = 0$, one has

$$v(0) = (0 : a'(0)b(0)/2 : * : *), \quad v'(0) = ((a'(0))^2b(0) : * : * : *),$$

so if $v(0)$ and $v'(0)$ are proportional then $(a'(0))^3(b(0))^2 = 0$. However, $a'(0) \neq 0$ since X is transversal to the line $\{x_0 = 0\}$ (hypothesis (1)).

Hence, $b(0) = a(0) = 0$ and the curve X passes through the point $(0 : 0 : 1)$. This contradicts hypothesis (4). \square

Lemma 4.8. *Suppose that $X \subset \mathbb{P}^2$ is a nodal curve with the following properties.*

- (1) X intersects transversally the line $\{x_0 = 0\}$.
- (2) Tangent lines to X at its intersection points with the line $\{x_0 = 0\}$, do not pass through the point $(0 : 1 : 0)$.
- (3) No bitangent to X passes through the point $(0 : 1 : 0)$.

Then the conormal variety $\mathcal{P}_X \subset \mathbb{P}^ \mathcal{T}_{\mathbb{P}^3}$ is smooth, lies in the open set where the birational isomorphism $\vartheta(\mathcal{P}_X) \subset \mathbb{P}^3$ is defined, and the restriction $\vartheta|_{\mathcal{P}_X} : \mathcal{P}_X \rightarrow \mathbb{P}^3$ is one-to-one onto its image.*

Proof. The first two assertions follow from hypotheses (1) and (2) as before, and again \mathcal{P}_X is smooth. To prove injectivity of $\vartheta|_{\mathcal{P}_X}$, represent \mathcal{P}_X as disjoint union of the following three subsets A , B and C :

$$\begin{aligned} A &= \{((x_0 : x_1 : x_2), (y_0 : y_1 : y_2)) \in \mathcal{P}_X : x_0 = 0\}, \\ B &= \{((x_0 : x_1 : x_2), (y_0 : y_1 : y_2)) \in \mathcal{P}_X : x_0 \neq 0, y_1 = 0\}, \\ C &= \{((x_0 : x_1 : x_2), (y_0 : y_1 : y_2)) \in \mathcal{P}_X : x_0 \neq 0, y_1 \neq 0\}. \end{aligned}$$

Now the required injectivity of $\vartheta|_{\mathcal{P}_X}$ is implied by the following chain of assertions.

1. ϑ is injective on A . Indeed, if $p = ((0 : x_1 : x_2), (y_0 : y_1 : y_2)) \in A$, then $\vartheta(p) = (0 : \frac{1}{2}x_1 : x_2 : 0)$. Thus, if $p_1, p_2 \in A$, then $\vartheta(p_1) = \vartheta(p_2)$ if and only if $\pi(p_1) = \pi(p_2)$, where $\pi : \mathcal{P}_X \rightarrow X$ is the natural projection. Since the line $\{x_0 = 0\}$ does not pass through the nodes of X (hypothesis (1)), this implies that $p_1 = p_2$.

2. $\vartheta(A) \cap \vartheta(B \cup C) = \emptyset$. Indeed, the z_0 coordinate is zero for any point in $\vartheta(A)$, and z_0 is non-zero for any point in $\vartheta(C)$, see (3.5). Thus, $\vartheta(A)$ is disjoint with $\vartheta(C)$. If $\vartheta(p) = \vartheta(q)$, where $p = ((0 : x_1 : x_2), (\cdot : \cdot : \cdot)) \in A$ and $q = ((\cdot : \cdot : \cdot), (y_0 : 0 : y_2)) \in B$, then

$$(0 : \frac{1}{2}x_1 : x_2 : 0) = (0 : y_0 : 0 : y_2),$$

whence $y_2 = 0$. Thus, the second component of the point $q \in \mathcal{P}_X$ is the line with homogeneous coordinates $(1 : 0 : 0)$, i.e., the line with equation $x_0 = 0$. This is, however, impossible since this line is not tangent to X , thanks to hypothesis (1).

3. ϑ is injective on B . Indeed, if $p = ((x_0 : x_1 : x_2), (y_0 : 0 : y_2)) \in B$, then $\vartheta(p) = (0 : y_0 : 0 : y_2)$. So, if the points $p_1 = (x_1, H_1)$ and $p_2 = (x_2, H_2)$ ($x_i \in X$, H_i is tangent to X at x_i) lie in B and if $\vartheta(p_1) = \vartheta(p_2)$, then $\ell_1 = \ell_2$ and the line $H = H_1 = H_2$ passes through the point $(0 : 1 : 0)$ and is tangent to X at two different points x_1 and x_2 ; this contradicts hypothesis (3).

4. $\vartheta(B) \cap \vartheta(C) = \emptyset$. Indeed the z_0 coordinate of $\vartheta(p)$ is zero if $p \in B$ and it is non-zero if $p \in C$.

5. ϑ is injective on C . This is implied by Proposition 3.5. \square

Proof of Proposition 4.6. Any smooth projective curve $C \subset \mathbb{P}^N$ can be birationally projected to a nodal plane curve $X \subset \mathbb{P}^2$; it is clear that C , being the normalization of X , is isomorphic to \mathcal{P}_X . For a general projective transformation $A: \mathbb{P}^2 \rightarrow \mathbb{P}^2$, the curve $X_1 = AX \subset \mathbb{P}^2$ satisfies the hypotheses of Lemmas 4.7 and 4.8, so the curve $\vartheta(\mathcal{P}_{X_1}) \subset \mathbb{P}^3$ is smooth, isomorphic to C , and Legendrian with respect to the contact structure defined by the formula (3.2) with $n = 2$ \square

Corollary 4.9 (from the proof). *If $C \subset \mathbb{P}^N$ is a smooth projective curve of degree d and genus g , then there exists a Legendrian curve $C' \subset \mathbb{P}^3$ such that C' is isomorphic to C and $\deg C' = 3d + 2g - 2$. In particular, if C is a smooth plane curve of degree d , then $\deg C' = d^2$.*

Proof. If $\deg C = d$ and genus of C equals g , then its general projection $X \subset \mathbb{P}^2$ has $\nu = (d-1)(d-2)/2 - g$ nodes, whence

$$\deg X^* = \deg(X')^* = d(d-1) - 2\nu = 2d + 2g - 2.$$

In the proof of Proposition 4.5 we found out that $\deg \mathcal{P}_{X'} = \deg X' + \deg(X')^*$, whence the formula. \square

5. CONCLUDING REMARKS

Although Legendrian subvarieties in odd-dimensional projective spaces abound, there exist osculating self-dual varieties (k -dimensional in \mathbb{P}^{2k+1}) that are not Legendrian with respect to any contact structure on \mathbb{P}^{2k+1} .

For $k = 1$, i.e., for the case of curves in \mathbb{P}^3 , it is easy to produce a family of examples.

Recall that a *monomial curve* in $C_{a,b,c} \subset \mathbb{P}^3$ is the closure of the set of points with homogeneous coordinates $(1 : t^a : t^b : t^c)$, where a, b, c are positive integers, $(a, b, c) = 1$, and $a < b < c$.

Proposition 5.1. *Any monomial curve in \mathbb{P}^3 is osculating self-dual. The monomial curve $C_{a,b,c} \subset \mathbb{P}^3$ is Legendrian with respect to an appropriate contact structure on \mathbb{P}^3 if and only if the sequence of exponents $(0, a, b, c)$ is symmetric, i.e., $(0, a, b, c) = (0, c - b, c - a, c)$.*

Proof. A Zariski open part of the curve $C_{a,b,c}$ can be (locally) parametrized by the formula $t \mapsto (v(t))$, where $v(t) = (1, t^a, t^b, t^c) \in \mathbb{C}^4$, $\mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4)$, $t \in \mathbb{C}$. Homogeneous coordinates of the osculating dual curve C^\vee are, up to signs, 3×3 minors of the matrix

$$\begin{pmatrix} 1 & t^a & t^b & t^c \\ 0 & at^{a-1} & bt^{b-1} & ct^{c-1} \\ 0 & a(a-1)t^{a-2} & b(b-1)t^{b-2} & c(c-1)t^{c-2} \end{pmatrix}$$

(of which the rows are $v(t)$, $v'(t)$, and $v''(t)$). A simple computation shows that $C_{a,b,c}^\vee$ is projectively equivalent to the curve that can be locally parametrized

as $t \mapsto (1 : t^{c-b} : t^{c-a} : t^c)$. After the linear automorphism that rearranges homogeneous coordinates in reverse order and the change of parameter $t = 1/s$, this dual curves becomes C ; this proves self-duality.

Now the curve $C_{a,b,c}$ is Legendrian if and only if there exists a non-degenerate skew-symmetric form B on \mathbb{C}^4 such that $B(v(t), v'(t)) = 0$ identically. If matrix of this bilinear form is $\|p_{ij}\|_{0 \leq i,j \leq 3}$, then

$$(5.1) \quad B(v(t), v'(t)) = ap_{01}t^{a-1} + bp_{02}t^{b-1} + cp_{03}t^{c-1} \\ + (b-a)p_{12}t^{a+b-1} + (c-a)p_{13}t^{a+c-1} + (c-b)p_{23}t^{b+c-1}.$$

If the sequence $(0, a, b, c)$ is not symmetric, then all the exponents in the right-hand side of (5.1) are different, so each p_{ij} is zero and the required contact structure does not exist. If, on the other hand, this sequence is symmetric, i.e., if $c = a + b$, then right-hand side of (5.1) is identically zero if and only if $cp_{03} + (b-a)p_{12} = 0$, so putting

$$B = \begin{pmatrix} 0 & 0 & 0 & a-b \\ 0 & 0 & c & 0 \\ 0 & -c & 0 & 0 \\ b-a & 0 & 0 & 0 \end{pmatrix}$$

one obtains a contact structure with respect to which the curve $C_{a,b,c}$ is Legendrian. \square

The following proposition provides an example in higher dimensions.

Proposition 5.2. *Denote by $V \subset \mathbb{P}^{2k+1}$, where $k \geq 2$ is an integer, the closure of the set of points $(v(t))$, $t \in \mathbb{C}^k$, where*

$$v(t) = (1, t_1, \dots, t_k, t_1^2, \dots, t_k^2, t_1^3 + \dots + t_k^3).$$

Then $\dim V = k$, $\dim_p \text{Osc}_p^2 V = 2k$ for general $p \in V$, V is osculating self-dual, but V is not Legendrian with respect to any contact structure on \mathbb{P}^{2n+1} .

Proof. One has

$$(5.2) \quad \begin{array}{lcl} v & = & (1, \quad t_1, \quad t_2, \quad \dots, \quad t_k, \quad t_1^2, \quad t_2^2, \quad \dots, \quad t_k^2, \quad t_1^3 + \dots + t_k^3), \\ \frac{\partial v}{\partial t_1} & = & (0, \quad 1, \quad 0, \quad \dots, \quad 0, \quad 2t_1, \quad 0, \quad \dots, \quad 0, \quad 3t_1^2), \\ \dots & & \dots \\ \frac{\partial v}{\partial t_k} & = & (0, \quad 0, \quad 0, \quad \dots, \quad 1, \quad 0, \quad 0, \quad \dots, \quad 2t_k, \quad 3t_k^2), \\ \frac{\partial^2 v}{\partial t_1^2} & = & (0, \quad 0, \quad 0, \quad \dots, \quad 0, \quad 2, \quad 0, \quad \dots, \quad 0, \quad 6t_1), \\ \dots & & \dots \\ \frac{\partial^2 v}{\partial t_k^2} & = & (0, \quad 0, \quad 0, \quad \dots, \quad 0, \quad 0, \quad 0, \quad \dots, \quad 2, \quad 6t_k) \end{array}$$

(other second partial derivatives of v are identically zero). Thus, for general t_1, \dots, t_k , dimension of the second osculating space is $2k$ indeed. Homogeneous coordinates of V^\vee are parametrized by $(2k+1) \times (2k+1)$ -minors of the $(2k+1) \times (2k+2)$ -matrix formed by the right-hand sides of (5.2). Direct

computation shows that these coordinates, up to non-zero constant factors, are

$$(1 : t_1 : \dots : t_k : t_1^2 : \dots : t_k^2 : t_1^3 + \dots + t_k^3 + P(t_1, \dots, t_k)),$$

where $P(t_1, \dots, t_k)$ is a linear combination of t_1, \dots, t_k and t_1^2, \dots, t_k^2 with constant coefficients. It is clear that this variety is projectively equivalent to V , so V is osculating self-dual.

Suppose now that V is Legendrian with respect to the contact structure corresponding to a skew-symmetric form B with matrix $\|p_{ij}\|$. Proposition 3.2 implies that $B(\partial v / \partial t_i, \partial v / \partial t_j) = 0$ identically for $1 \leq i < j \leq k$; substituting the expressions from (5.2), one obtains that $p_{ij} = p_{i,k+j} = p_{k+i,k+j} = 0$ for $1 \leq i, j \leq k$ and $p_{i,2k+1} = p_{k+i,2k+1} = 0$ for $1 \leq i \leq k$. Similarly, since $B(v, \partial v / \partial t_j) = 0$ identically for $1 \leq i \leq k$, one obtains, taking into account that $p_{\alpha\beta} = 0$ for $1 \leq \alpha, \beta \leq 2k$, that $p_{0,2k+1} = 0$. These vanishing implies that $\det \|p_{ij}\| = 0$, which contradicts the non-degeneracy of the form B . \square

For $k = 2$, the surface $V \subset \mathbb{P}^5$ is projectively equivalent to Togliatti's surface (II) (see [Tog29, p. 261]).

REFERENCES

- [Bry82] Robert L. Bryant, *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Differential Geom. **17** (1982), no. 3, 455–473. MR 679067 (84a:53062)
- [Hol26] Temple Rice Hollcroft, *Conditions for self dual curves*, Ann. of Math. (2) **27** (1926), no. 3, 258–270. MR 1502731
- [Kle86] Steven L. Kleiman, *Tangency and duality*, Proceedings of the 1984 Vancouver conference in algebraic geometry (Providence, RI), CMS Conf. Proc., vol. 6, Amer. Math. Soc., 1986, pp. 163–225. MR 846021 (87i:14046)
- [Pie77] Ragni Piene, *Numerical characters of a curve in projective n -space*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 475–495. MR 0506323 (58 #22095)
- [Seg07] C. Segre, *Su una classe di superficie degli iperspazi legata colle equazioni lineari alle derivate parziali di 2° ordine*, Torino Atti **42** (1907), 1047–1079 (Italian).
- [Tog29] E. G. Togliatti, *Alcuni esempi di superficie algebriche degli iperspazi che rappresentano un'equazione di Laplace*, Comment. Math. Helv. **1** (1929), 255–272 (Italian).
- [Val06] Jean Vallès, *Variétés de type Togliatti*, C. R. Math. Acad. Sci. Paris **343** (2006), no. 6, 411–414. MR 2259882 (2007f:14034)

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, MOSCOW, RUSSIA
E-mail address: lvovski@gmail.com